Transient behavior of 2x2 matrix ODE models

Some thoughts on Neubert&Caswell 1997[1] and Neubert et al. 2004[2]

Cora Kohlmeier

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[1] Alternatives to Resilience for Measuring the Responses of Ecological Systems to Perturbations

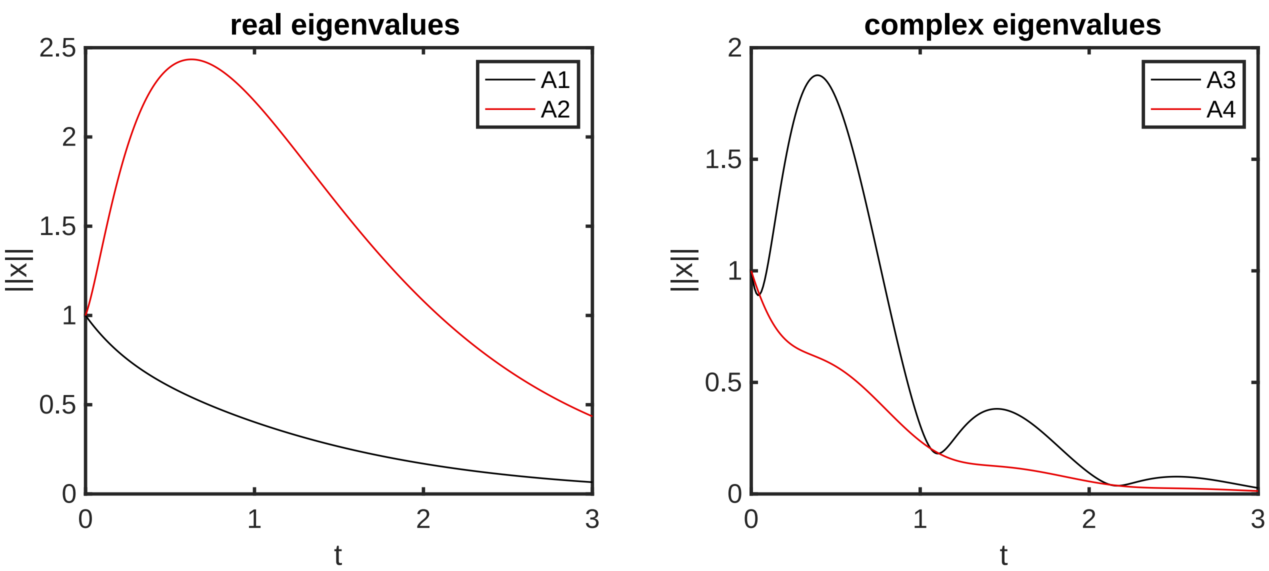
[2] Reactivity and transient dynamics of predator–prey

Linear model

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0 \qquad\qquad \text{Solution} \quad x(t) = e^{At}x_0$$

Asymptotic stable systems

- fixed point (0,0)
- all eigenvalues of A have negative real parts



$A, A_1, \ldots, A_4 \in \mathbf{R}^{2x2}, x \in \mathbf{R}^2$

Transient behavior

- Initial behavior of the system depends on inital pertubation
- 2 characteristic quantities
  - Maximum amplification - the overall maximum deflection from fixed point
  - Reactivity – the overall maximum initial slope

Maximum amplification ρ

Maximum amplification for initial angle Φ

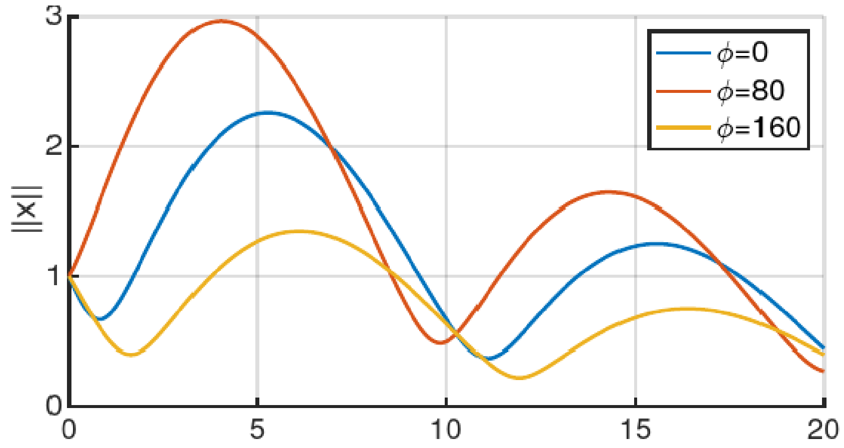
$$\rho(\Phi) \equiv \max_{t \geq 0} \frac{\|x(t)\|}{\|x_0(\Phi)\|}$$

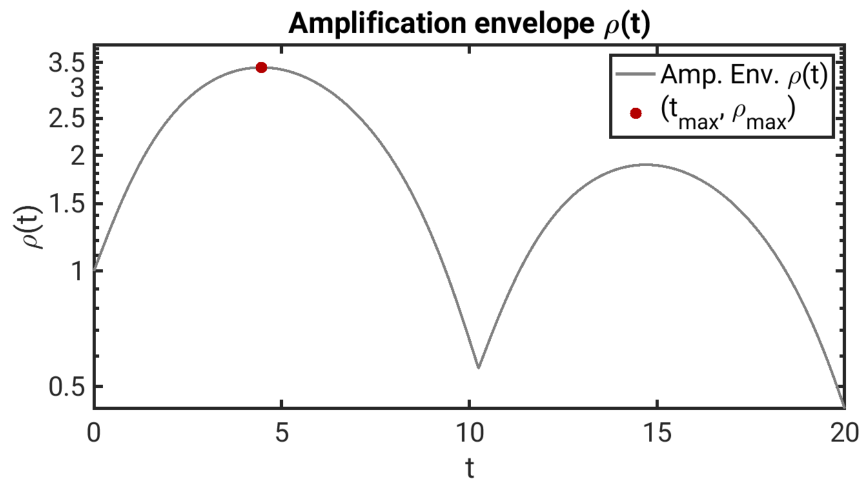
Maximum amplification at time t

$$\rho(t) \equiv \max_{\|x_0\| \neq 0} \frac{\|x(t)\|}{\|x_0\|} = \| e^{At} \|$$

Maximum amplification

$$\rho_{max} \equiv \max_{t \geq 0} \rho(t)$$





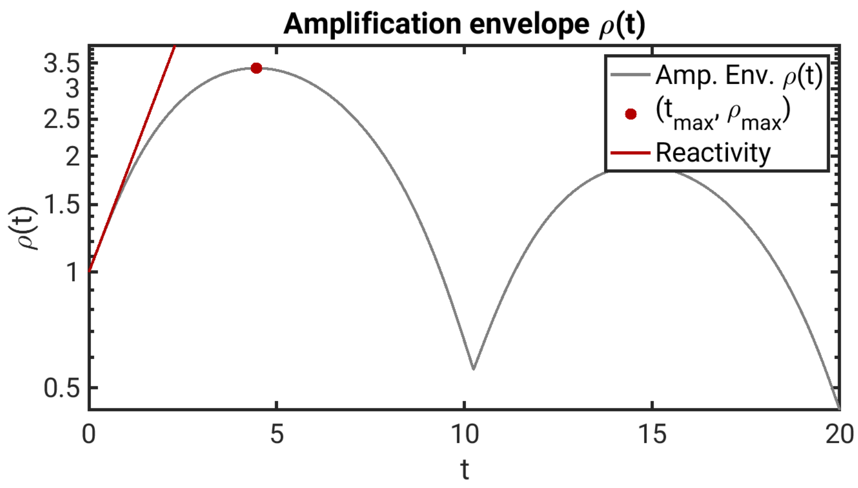
Thanks to Christoph Feenders for coding the amplification envelope

Reactivity ν_0

Maximum amplification rate at fixed point (0,0) over all inital pertubations

$$\nu_0 \equiv \max_{\|x_0\|\neq 0}\left[\left(\frac{1}{\|x\|}\frac{d\|x\|}{dt}\right)\Big|_{t=0}\right] = \max_{\|x_0\|\neq 0}\left[\frac{x_0^T H x_0}{x_0^T x_0}\right] = \lambda_1(H(A)) > 0$$

- maximum initial slope normed by pertubation
- transient property, no information about long term behavior



Reactivity of linear 2x2 system

Assume

$$A = \begin{pmatrix} a & d \\ c & d \end{pmatrix} \implies H = \frac{A + A^T}{2} = \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix}$$

Reactivity

$$\nu_0 = \lambda_1(H) = \frac{1}{2}\left(a + d + \sqrt{(a-d)^2 + (b+c)^2}\right)$$

$H(A) = \frac{A+A^T}{2}$ Hermitian part of A

$\frac{x_0^T H x_0}{x_0^T x_0}$ Rayleigh quotient, maximized by largest eigenvalue $\lambda_1(H)$ of H

Questions

- Does the maximum amplification depend on the type of eigenvalues (real, complex)?
- Does the maximum amplification depend on angle of pertubation?
- Does the maximum amplification depend on the eigen direction?
- Are eigen direction and maximum direction related?
- What's about reactivity?

Why is this important ?

- Mostly transient behavior is observed in nature
  - due to duration of projects
  - steady state cannot be reached
- Transient behavior determines measures to be taken
  - quick response in f.e. epidemiology is required

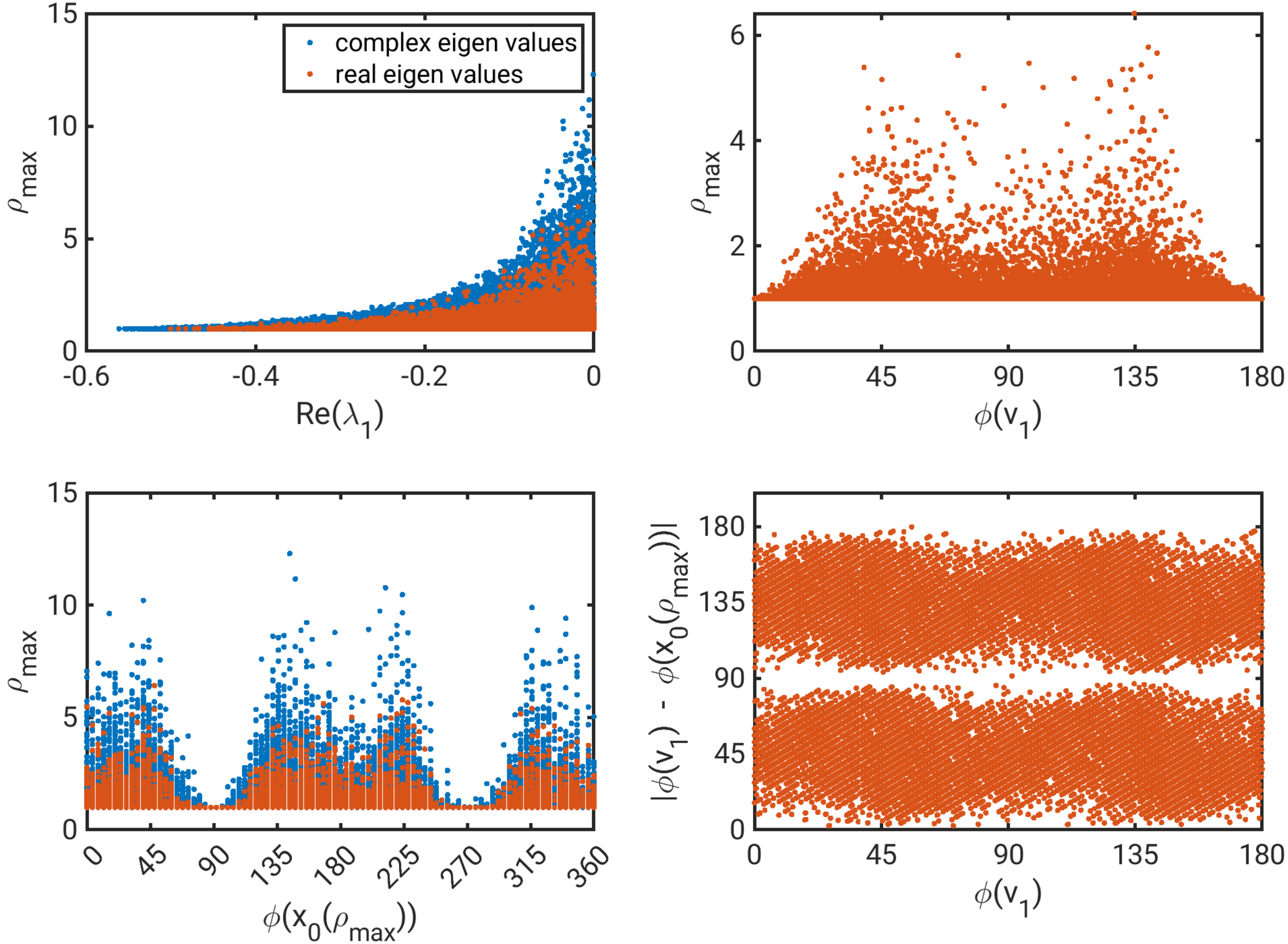
Simulation set up

$$\frac{dx}{dt} = Ax, \quad x(0) = (\cos\phi, \sin\phi)$$

- $|a_{ij}| \leq 1$
- $\approx$ 90000 stable[3] and reactive systems out of 10^6 randomly chosen 2x2 matrices with
- ϕ in steps of 4°
- ODE solver: Matlab ode45

[3] negative real part of eigen values, $\approx$ 22000 real eigen values

Simulation ρ_{max}



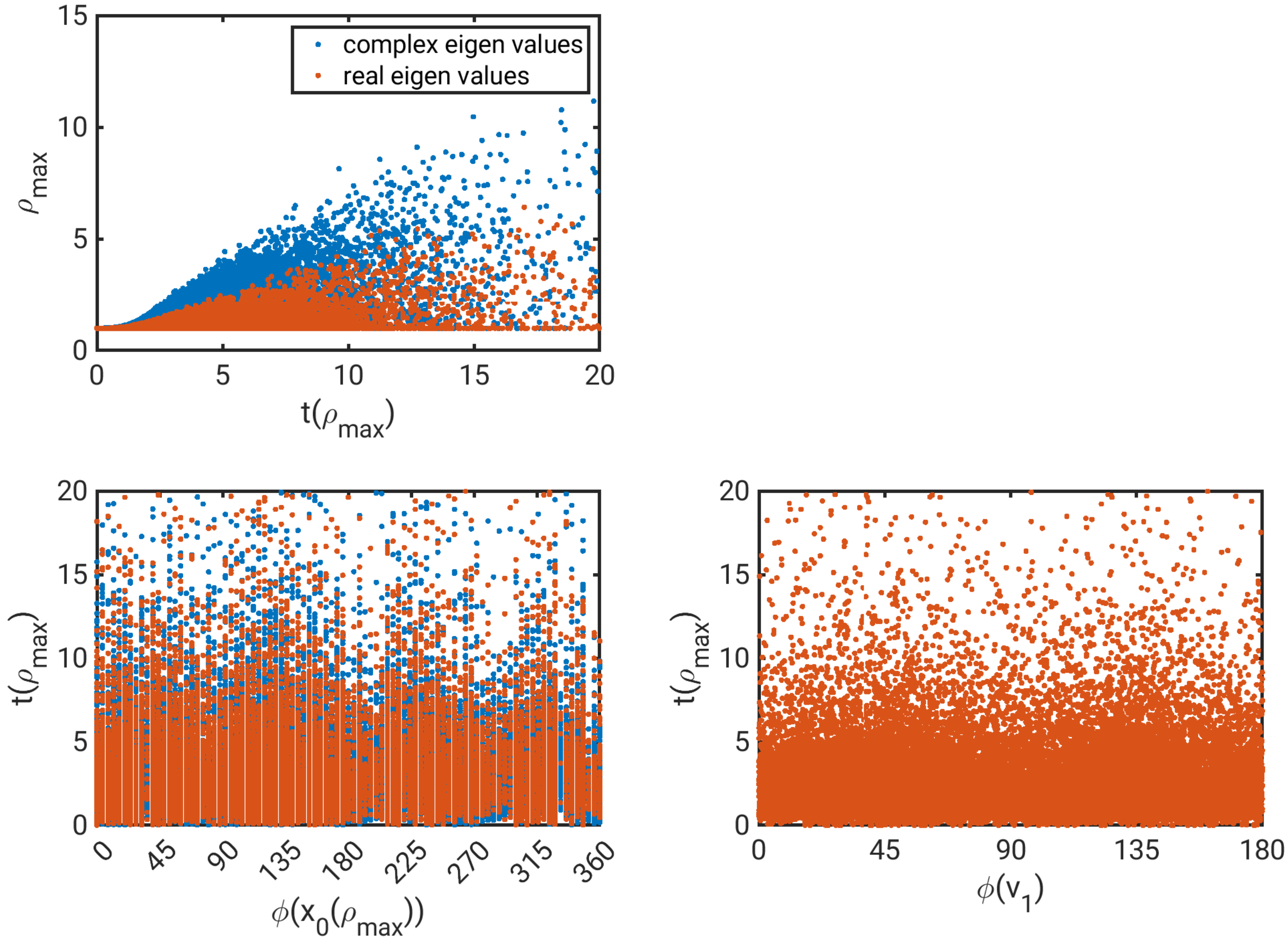
Observations 1

- amplification increases with increasing largest EV
- systems with complex EV can have higher amplifications due to superposition of amplification by pertubation and oscillations
- few matrices have largest amplification around 90°and 270°pertubation and their amplification is relatively low
- amplification is lower around 90°resp. 180°ED and higher in between
- tendency for larger amplification if ED is 45°or 135 °
- nearly no system has largest amplification with initial pertubation perpendicular or parallel to ED

EV: largest eigen vector
ED: direction of largest eigen vector

DI: distance between direction of largest eigen vector and direction of maximum amplification

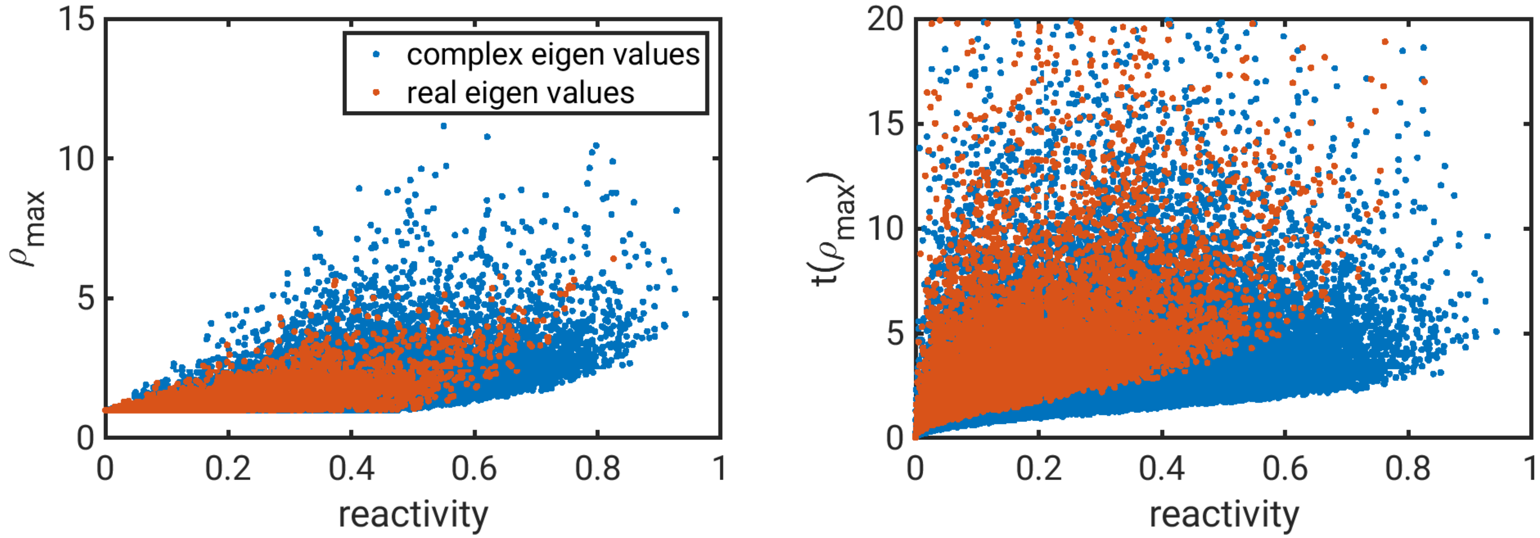
Simulation Time of ρ_{max}



Observations 2

- time and magnitude of maximum amplification are correlated
- no obvious relation between angle of maximum amplification and time of maximum amplification
- no obvious relation direction of largest eigen vector and time of maximum amplification

Simulation Reactivity



Observations 3

- more matrices with complex eigenvalues reaches high reactivity
- reactivity and maximum amplification are correlated
- reactivity and maximum amplification of matrices with real eigenvalues tends to be smaller
- time of reaching ρ_{max} tends to be larger for matrices with real eigenvalues

Conclusion and open questions

- Reactivity and maximum amplification are correlated, but may belong to different initial pertubations
- no obvious connection of initial pertubation angle and eigen value to time of maximum pertubation
- Quantification of observations missing
- Relationship of the angle of initial pertubation and the maximum pertubation missing

Literature

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